This example is set up in American Typewriter.

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\usepackage[no-math]{fontspec}
\setmainfont[Mapping=tex-text]{American Typewriter}
\usepackage[defaultmathsizes]{mathastext}
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Typeset with mathastext 1.15d (2012/10/13).
(compiled with XeLaTeX)

Theorem 1. Let there be given indeterminates u_i, v_i, k_i, x_i, y_i, l_i, for $1 \leqslant i \leqslant n$. We define the following $n \times n$ matrices

$$U_n = \begin{pmatrix} u_1 & u_2 & \dots & u_n \\ k_1 v_1 & k_2 v_2 & \dots & k_n v_n \\ k_1^2 u_1 & k_2^2 u_2 & \dots & k_n^2 u_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \qquad V_n = \begin{pmatrix} v_1 & v_2 & \dots & v_n \\ k_1 u_1 & k_2 u_2 & \dots & k_n u_n \\ k_1^2 v_1 & k_2^2 v_2 & \dots & k_n^2 v_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \tag{1}$$

where the rows contain alternatively u's and v's. Similarly:

$$X_n = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ l_1 y_1 & l_2 y_2 & \dots & l_n y_n \\ l_1^2 x_1 & l_2^2 x_2 & \dots & l_n^2 x_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \qquad Y_n = \begin{pmatrix} y_1 & y_2 & \dots & y_n \\ l_1 x_1 & l_2 x_2 & \dots & l_n x_n \\ l_1^2 y_1 & l_2^2 y_2 & \dots & l_n^2 y_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \tag{2}$$

There holds

$$\det_{1 \leqslant i,j \leqslant n} \left(\frac{u_i y_j - v_i x_j}{l_j - k_i} \right) = \frac{1}{\prod_{i,j}(l_j - k_i)} \begin{vmatrix} U_n & X_n \\ V_n & Y_n \end{vmatrix}_{2n \times 2n} \tag{3}$$

Proof. Let A, B, C, D be $n \times n$ matrices, with A and C invertible. Using $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & 0 \\ 0 & C \end{smallmatrix}\right) \left(\begin{smallmatrix} I & A^{-1}B \\ I & C^{-1}D \end{smallmatrix}\right)$ we obtain

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\,|C|\,|C^{-1}D - A^{-1}B| \tag{4}$$

where vertical bars denote determinants. Let $d(u) = \operatorname{diag}(u_1, \dots, u_n)$ and $p_u = \prod_{1 \leqslant i \leqslant n} u_i$. We define similarly $d(v)$, $d(x)$, $d(y)$ and p_v, p_x, p_y. From the previous identity we get

$$\begin{aligned} \begin{vmatrix} Ad(u) & Bd(x) \\ Cd(v) & Dd(y) \end{vmatrix} &= |A|\,|C|\; p_u p_v \left| d(v)^{-1} C^{-1} D d(y) - d(u)^{-1} A^{-1} B d(x) \right| \\ &= |A|\,|C| \left| d(u) C^{-1} D d(y) - d(v) A^{-1} B d(x) \right| \end{aligned} \tag{5}$$

The special case A = C, B = D, gives

$$\begin{vmatrix} Ad(u) & Bd(x) \\ Ad(v) & Bd(y) \end{vmatrix}_{2n \times 2n} = \det(A)^2 \det_{1 \leqslant i,j \leqslant n} ((u_i y_j - v_i x_j)(A^{-1}B)_{ij}) \tag{6}$$

Let $W(k)$ be the Vandermonde matrix with rows $(1 \dots 1)$, $(k_1 \dots k_n)$, $(k_1^2 \dots k_n^2)$, ..., and $\Delta(k) = \det W(k)$ its determinant. Let

$$K(t) = \prod_{1 \leqslant m \leqslant n} (t - k_m) \tag{7}$$

and let C be the $n \times n$ matrix $(c_{im})_{1\leqslant i,m\leqslant n}$, where the c_{im}'s are defined by the partial fraction expansions:

$$1 \leqslant i \leqslant n \qquad \frac{t^{i-1}}{K(t)} = \sum_{1\leqslant m\leqslant n} \frac{c_{im}}{t - k_m} \tag{8}$$

We have the two matrix equations:

$$C = W(k)\,\mathrm{diag}(K'(k_1)^{-1}, \ldots, K'(k_n)^{-1}) \tag{9a}$$

$$C \cdot \left(\frac{1}{l_j - k_m}\right)_{1\leqslant m,j\leqslant n} = W(l)\,\mathrm{diag}(K(l_1)^{-1}, \ldots, K(l_n)^{-1}) \tag{9b}$$

This gives the (well-known) identity:

$$\left(\frac{1}{l_j - k_m}\right)_{1\leqslant m,j\leqslant n} = \mathrm{diag}(K'(k_1), \ldots, K'(k_n))W(k)^{-1}W(l)\,\mathrm{diag}(K(l_1)^{-1}, \ldots, K(l_n)^{-1}) \tag{10}$$

We can thus rewrite the determinant we want to compute as:

$$\left|\frac{u_i y_j - v_i x_j}{l_j - k_i}\right|_{1\leqslant i,j\leqslant n} = \prod_m K'(k_m) \prod_j K(l_j)^{-1} \left|(u_i y_j - v_i x_j)(W(k)^{-1}W(l))_{ij}\right|_{n\times n} \tag{11}$$

We shall now make use of (6) with $A = W(k)$ and $B = W(l)$.

$$\begin{aligned}\left|\frac{u_i y_j - v_i x_j}{l_j - k_i}\right|_{1\leqslant i,j\leqslant n} &= \Delta(k)^{-2} \prod_m K'(k_m) \prod_j K(l_j)^{-1} \begin{vmatrix} W(k)d(u) & W(l)d(x) \\ W(k)d(v) & W(l)d(y) \end{vmatrix} \\ &= \frac{(-1)^{\frac{n(n-1)}{2}}}{\prod_{i,j}(l_j - k_i)} \begin{vmatrix} W(k)d(u) & W(l)d(x) \\ W(k)d(v) & W(l)d(y) \end{vmatrix}_{2n\times 2n}\end{aligned} \tag{12}$$

The sign $(-1)^{n(n-1)/2} = (-1)^{[\frac{n}{2}]}$ is the signature of the permutation which exchanges rows i and $n+i$ for $i = 2, 4, \ldots, 2[\frac{n}{2}]$ and transforms the determinant on the right-hand side into $\begin{vmatrix} U_n & X_n \\ V_n & Y_n \end{vmatrix}$. This concludes the proof. $\square$