This example is set up in ECF Tall Paul (with Symbol font). It uses:

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\DeclareFontFamily{T1}{ftp}{}
\DeclareFontShape{T1}{ftp}{m}{n}{
   <->s*[1.4] ftpmw8t
}{} % increase size by factor 1.4
\renewcommand\familydefault{ftp} % emerald package
\usepackage[symbol]{mathastext}
\let\infty\inftypsy
```

Typeset with mathastext 1.15d (2012/10/13).

To illustrate some Hilbert Space properties of the co-Poisson summation, we will assume $K = \mathbb{Q}$. The components (a_v) of an adele a are written a_p at finite places and a_r at the real place. We have an embedding of the Schwartz space of test-functions on $\mathbb{R}$ into the Bruhat-Schwartz space on $\mathbb{A}$ which sends $\psi(x)$ to $\phi(a) = \prod_p 1_{|a_p|_p\leq 1}(a_p) \cdot \psi(a_r)$, and we write $\Xi'_{\mathbb{R}}(g)$ for the distribution on $\mathbb{R}$ thus obtained from $\Xi'(g)$ on $\mathbb{A}$.

Theorem 1. Let g be a compact Bruhat-Schwartz function on the ideles of $\mathbb{Q}$. The co-Poisson summation $\Xi'_{\mathbb{R}}(g)$ is a square-integrable function (with respect to the Lebesgue measure). The $L^2(\mathbb{R})$ function $\Xi'_{\mathbb{R}}(g)$ is equal to the constant $-\int_{\mathbb{A}^\times} g(v)|v|^{-1/2}d^*v$ in a neighborhood of the origin.

Proof. We may first, without changing anything to $\Xi'_{\mathbb{R}}(g)$, replace g with its average under the action of the finite unit ideles, so that it may be assumed invariant. Any such compact invariant g is a finite linear combination of suitable multiplicative translates of functions of the type $g(v) = \prod_p 1_{|v_p|_p=1}(v_p) \cdot f(v_r)$ with $f(t)$ a smooth compactly supported function on $\mathbb{R}^\times$, so that we may assume that g has this form. We claim that:

$$\int_{\mathbb{A}^\times} |\phi(v)| \sum_{q\in\mathbb{Q}^\times} |g(qv)| \sqrt{|v|}\, d^*v < \infty$$

Indeed $\sum_{q\in\mathbb{Q}^\times} |g(qv)| = |f(|v|)| + |f(-|v|)|$ is bounded above by a multiple of $|v|$. And $\int_{\mathbb{A}^\times} |\phi(v)||v|^{3/2} d^*v < \infty$ for each Bruhat-Schwartz function on the adeles (basically, from $\prod_p (1-p^{-3/2})^{-1} < \infty$). So

$$\Xi'(g)(\phi) = \sum_{q\in\mathbb{Q}^\times} \int_{\mathbb{A}^\times} \phi(v) g(qv) \sqrt{|v|}\, d^*v - \int_{\mathbb{A}^\times} \frac{g(v)}{\sqrt{|v|}} d^*v \int_{\mathbb{A}} \phi(x)\, dx$$

$$\Xi'(g)(\phi) = \sum_{q\in\mathbb{Q}^\times} \int_{\mathbb{A}^\times} \phi(v/q) g(v) \sqrt{|v|}\, d^*v - \int_{\mathbb{A}^\times} \frac{g(v)}{\sqrt{|v|}} d^*v \int_{\mathbb{A}} \phi(x)\, dx$$

Let us now specialize to $\phi(a) = \prod_p 1_{|a_p|_p\leq 1}(a_p) \cdot \psi(a_r)$. Each integral can be evaluated as an infinite product. The finite places contribute 0 or 1 according to whether $q \in \mathbb{Q}^\times$ satisfies $|q|_p < 1$ or not. So only the inverse integers $q = 1/n$, $n \in \mathbb{Z}$, contribute:

$$\Xi'_{\mathbb{R}}(g)(\psi) = \sum_{n\in\mathbb{Z}^\times} \int_{\mathbb{R}^\times} \psi(nt) f(t) \sqrt{|t|} \frac{dt}{2|t|} - \int_{\mathbb{R}^\times} \frac{f(t)}{\sqrt{|t|}} \frac{dt}{2|t|} \int_{\mathbb{R}} \psi(x)\, dx$$

We can now revert the steps, but this time on $\mathbb{R}^\times$ and we get:

$$\Xi'_{\mathbb{R}}(g)(\psi) = \int_{\mathbb{R}^\times} \psi(t) \sum_{n\in\mathbb{Z}^\times} \frac{f(t/n)}{\sqrt{|n|}} \frac{dt}{2\sqrt{|t|}} - \int_{\mathbb{R}^\times} \frac{f(t)}{\sqrt{|t|}} \frac{dt}{2|t|} \int_{\mathbb{R}} \psi(x)\,dx$$

Let us express this in terms of $\alpha(y) = (f(y) + f(-y))/2\sqrt{|y|}$:

$$\Xi'_{\mathbb{R}}(g)(\psi) = \int_{\mathbb{R}} \psi(y) \sum_{n\geq 1} \frac{\alpha(y/n)}{n} dy - \int_0^\infty \frac{\alpha(y)}{y} dy \int_{\mathbb{R}} \psi(x)\,dx$$

So the distribution $\Xi'_{\mathbb{R}}(g)$ is in fact the even smooth function

$$\Xi'_{\mathbb{R}}(g)(y) = \sum_{n\geq 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy$$

As $\alpha(y)$ has compact support in $\mathbb{R}\setminus\{0\}$, the summation over $n \geq 1$ contains only vanishing terms for $|y|$ small enough. So $\Xi'_{\mathbb{R}}(g)$ is equal to the constant $-\int_0^\infty \frac{\alpha(y)}{y} dy = -\int_{\mathbb{R}^\times} \frac{f(y)}{\sqrt{|y|}} \frac{dy}{2|y|} = -\int_{\mathbb{A}^\times} g(t)/\sqrt{|t|}\, d^*t$ in a neighborhood of 0. To prove that it is L^2, let $\beta(y)$ be the smooth compactly supported function $\alpha(1/y)/2|y|$ of $y \in \mathbb{R}$ ($\beta(0) = 0$). Then ($y \neq 0$):

$$\Xi'_{\mathbb{R}}(g)(y) = \sum_{n\in\mathbb{Z}} \frac{1}{|y|} \beta(\frac{n}{y}) - \int_{\mathbb{R}} \beta(y)\,dy$$

From the usual Poisson summation formula, this is also:

$$\sum_{n\in\mathbb{Z}} \gamma(ny) - \int_{\mathbb{R}} \beta(y)\,dy = \sum_{n\neq 0} \gamma(ny)$$

where $\gamma(y) = \int_{\mathbb{R}} \exp(i2\pi yw)\beta(w)\,dw$ is a Schwartz rapidly decreasing function. From this formula we deduce easily that $\Xi'_{\mathbb{R}}(g)(y)$ is itself in the Schwartz class of rapidly decreasing functions, and in particular it is is square-integrable. □

It is useful to recapitulate some of the results arising in this proof:

Theorem 2. Let g be a compact Bruhat-Schwartz function on the ideles of $\mathbb{Q}$. The co-Poisson summation $\mathcal{E}'_{\mathbb{R}}(g)$ is an even function on $\mathbb{R}$ in the Schwartz class of rapidly decreasing functions. It is constant, as well as its Fourier Transform, in a neighborhood of the origin. It may be written as

$$\mathcal{E}'_{\mathbb{R}}(g)(y) = \sum_{n\geq 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy$$

with a function $\alpha(y)$ smooth with compact support away from the origin, and conversely each such formula corresponds to the co-Poisson summation $\mathcal{E}'_{\mathbb{R}}(g)$ of a compact Bruhat-Schwartz function on the ideles of $\mathbb{Q}$. The Fourier transform $\int_{\mathbb{R}} \mathcal{E}'_{\mathbb{R}}(g)(y) \exp(i2\pi wy)\, dy$ corresponds in the formula above to the replacement $\alpha(y) \mapsto \alpha(1/y)/|y|$.

Everything has been obtained previously.