This example is set up in GNU FreeFont Serif for the text, GNU FreeFont Sans for the letters in math mode, Latin Modern Sans for the Greek letters, and Computer Modern for the large symbols and delimiters.

```
\usepackage[no-math]{fontspec}
\setmainfont[ExternalLocation,
             Mapping=tex-text,
             BoldFont=FreeSerifBold,
             ItalicFont=FreeSerifItalic,
         BoldItalicFont=FreeSerifBoldItalic]{FreeSerif}
\setsansfont[ExternalLocation,
             Mapping=tex-text,
             BoldFont=FreeSansBold,
             ItalicFont=FreeSansOblique,
             BoldItalicFont=FreeSansBoldOblique,
             Scale=MatchLowercase]{FreeSans}
\renewcommand{\familydefault}{lmss}
\usepackage[LGRgreek,defaultmathsizes,noasterisk]{mathastext}
\renewcommand{\familydefault}{\sfdefault}
\Mathastext
\let\varphi\phi % no 'var' phi in LGR encoding
\renewcommand{\familydefault}{\rmdefault}
```

Typeset with mathastext 1.15d (2012/10/13).
(compiled with XƎLaTeX)

To illustrate some Hilbert Space properties of the co-Poisson summation, we will assume $K = \mathbf{Q}$. The components (a_v) of an adele a are written a_p at finite places and a_r at the real place. We have an embedding of the Schwartz space of test-functions on $\mathbf{R}$ into the Bruhat-Schwartz space on $\mathbf{A}$ which sends $\psi(x)$ to $\varphi(a) = \prod_p \mathbf{1}_{|a_p|_p \leqslant 1}(a_p) \cdot \psi(a_r)$, and we write $E'_{\mathbf{R}}(g)$ for the distribution on $\mathbf{R}$ thus obtained from $E'(g)$ on $\mathbf{A}$.

Theorem 1. *Let g be a compact Bruhat-Schwartz function on the ideles of $\mathbf{Q}$. The co-Poisson summation $E'_{\mathbf{R}}(g)$ is a square-integrable function (with respect to the Lebesgue measure). The $L^2(\mathbf{R})$ function $E'_{\mathbf{R}}(g)$ is equal to the constant $-\int_{\mathbf{A}^\times} g(v)|v|^{-1/2} d^*v$ in a neighborhood of the origin.*

Proof. We may first, without changing anything to $E'_{\mathbf{R}}(g)$, replace g with its average under the action of the finite unit ideles, so that it may be assumed invariant. Any such compact invariant g is a finite linear combination of suitable multiplicative translates of functions of the type $g(v) = \prod_p \mathbf{1}_{|v_p|_p=1}(v_p) \cdot f(v_r)$ with $f(t)$ a smooth compactly supported function on $\mathbf{R}^\times$, so that we may assume that g has this form. We claim that:

$$\int_{\mathbf{A}^\times} |\varphi(v)| \sum_{q \in \mathbf{Q}^\times} |g(qv)| \sqrt{|v|}\, d^*v < \infty$$

Indeed $\sum_{q\in\mathbf{Q}^\times} |g(qv)| = |f(|v|)| + |f(-|v|)|$ is bounded above by a multiple of $|v|$. And $\int_{\mathbf{A}^\times} |\varphi(v)||v|^{3/2}\, d^*v < \infty$ for each Bruhat-Schwartz function on the adeles (basically, from $\prod_p (1 - p^{-3/2})^{-1} < \infty$). So

$$E'(g)(\varphi) = \sum_{q\in\mathbf{Q}^\times} \int_{\mathbf{A}^\times} \varphi(v) g(qv) \sqrt{|v|}\, d^*v - \int_{\mathbf{A}^\times} \frac{g(v)}{\sqrt{|v|}} d^*v \int_{\mathbf{A}} \varphi(x)\, dx$$

$$E'(g)(\varphi) = \sum_{q\in\mathbf{Q}^\times} \int_{\mathbf{A}^\times} \varphi(v/q) g(v) \sqrt{|v|}\, d^*v - \int_{\mathbf{A}^\times} \frac{g(v)}{\sqrt{|v|}} d^*v \int_{\mathbf{A}} \varphi(x)\, dx$$

Let us now specialize to $\varphi(a) = \prod_p \mathbf{1}_{|a_p|_p \leqslant 1}(a_p) \cdot \psi(a_r)$. Each integral can be evaluated as an infinite product. The finite places contribute 0 or 1 according to whether $q \in \mathbf{Q}^\times$ satisfies $|q|_p < 1$ or not. So only the inverse integers $q = 1/n$, $n \in \mathbf{Z}$, contribute:

$$E'_{\mathbf{R}}(g)(\psi) = \sum_{n\in\mathbf{Z}^\times} \int_{\mathbf{R}^\times} \psi(nt) f(t) \sqrt{|t|} \frac{dt}{2|t|} - \int_{\mathbf{R}^\times} \frac{f(t)}{\sqrt{|t|}} \frac{dt}{2|t|} \int_{\mathbf{R}} \psi(x)\, dx$$

We can now revert the steps, but this time on $\mathbf{R}^\times$ and we get:

$$E'_{\mathbf{R}}(g)(\psi) = \int_{\mathbf{R}^\times} \psi(t) \sum_{n\in\mathbf{Z}^\times} \frac{f(t/n)}{\sqrt{|n|}} \frac{dt}{2\sqrt{|t|}} - \int_{\mathbf{R}^\times} \frac{f(t)}{\sqrt{|t|}} \frac{dt}{2|t|} \int_{\mathbf{R}} \psi(x)\, dx$$

Let us express this in terms of $\alpha(y) = (f(y) + f(-y))/2\sqrt{|y|}$:

$$E'_{\mathbf{R}}(g)(\psi) = \int_{\mathbf{R}} \psi(y) \sum_{n\geqslant 1} \frac{\alpha(y/n)}{n} dy - \int_0^\infty \frac{\alpha(y)}{y} dy \int_{\mathbf{R}} \psi(x)\, dx$$

So the distribution $E'_{\mathbf{R}}(g)$ is in fact the even smooth function

$$E'_{\mathbf{R}}(g)(y) = \sum_{n\geqslant 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy$$

As $\alpha(y)$ has compact support in $\mathbf{R} \setminus \{0\}$, the summation over $n \geqslant 1$ contains only vanishing terms for $|y|$ small enough. So $E'_{\mathbf{R}}(g)$ is equal to the constant $-\int_0^\infty \frac{\alpha(y)}{y} dy = -\int_{\mathbf{R}^\times} \frac{f(y)}{\sqrt{|y|}} \frac{dy}{2|y|} = -\int_{\mathbf{A}^\times} g(t)/\sqrt{|t|}\, d^*t$ in a neighborhood of 0. To prove that it is L^2, let $\beta(y)$ be the smooth compactly supported function $\alpha(1/y)/2|y|$ of $y \in \mathbf{R}$ ($\beta(0) = 0$). Then ($y \neq 0$):

$$E'_{\mathbf{R}}(g)(y) = \sum_{n\in\mathbf{Z}} \frac{1}{|y|} \beta\left(\frac{n}{y}\right) - \int_{\mathbf{R}} \beta(y)\, dy$$

From the usual Poisson summation formula, this is also:

$$\sum_{n\in\mathbf{Z}} \gamma(ny) - \int_{\mathbf{R}} \beta(y)\, dy = \sum_{n\neq 0} \gamma(ny)$$

where $\gamma(y) = \int_{\mathbf{R}} \exp(i\, 2\pi yw) \beta(w)\, dw$ is a Schwartz rapidly decreasing function. From this formula we deduce easily that $E'_{\mathbf{R}}(g)(y)$ is itself in the Schwartz class of rapidly decreasing functions, and in particular it is is square-integrable. □

It is useful to recapitulate some of the results arising in this proof:

Theorem 2. *Let* g *be a compact Bruhat-Schwartz function on the ideles of* $\mathbf{Q}$. *The co-Poisson summation* $E'_{\mathbf{R}}(g)$ *is an even function on* $\mathbf{R}$ *in the Schwartz class of rapidly decreasing functions. It is constant, as well as its Fourier Transform, in a neighborhood of the origin. It may be written as*

$$E'_{\mathbf{R}}(g)(y) = \sum_{n\geqslant 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy$$

with a function $\alpha(y)$ *smooth with compact support away from the origin, and conversely each such formula corresponds to the co-Poisson summation* $E'_{\mathbf{R}}(g)$ *of a compact Bruhat-Schwartz function on the ideles of* $\mathbf{Q}$. *The Fourier transform* $\int_{\mathbf{R}} E'_{\mathbf{R}}(g)(y) \exp(i2\pi wy)\, dy$ *corresponds in the formula above to the replacement* $\alpha(y) \mapsto \alpha(1/y)/|y|$.

Everything has been obtained previously.