This example is set up in Zapf Chancery.

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\DeclareFontFamily{T1}{pzc}{}
\DeclareFontShape{T1}{pzc}{mb}{it}{<->s*[1.2] pzcmi8t}{}
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\usepackage{chancery} % = \renewcommand{\rmdefault}{pzc}
\renewcommand\shapedefault\itdefault
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\usepackage[defaultmathsizes]{mathastext}
\linespread{1.05}
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Typeset with mathastext 1.15d (2012/10/13).

Theorem 1. *Let there be given indeterminates* u_i, v_i, k_i, x_i, y_i, l_i, *for* $1 \leqslant i \leqslant n$. *We define the following* $n \times n$ *matrices*

$$\mathcal{U}_n = \begin{pmatrix} u_1 & u_2 & \dots & u_n \\ k_1 v_1 & k_2 v_2 & \dots & k_n v_n \\ k_1^2 u_1 & k_2^2 u_2 & \dots & k_n^2 u_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \qquad \mathcal{V}_n = \begin{pmatrix} v_1 & v_2 & \dots & v_n \\ k_1 u_1 & k_2 u_2 & \dots & k_n u_n \\ k_1^2 v_1 & k_2^2 v_2 & \dots & k_n^2 v_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \tag{1}$$

where the rows contain alternatively u's and v's. Similarly:

$$\mathcal{X}_n = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ l_1 y_1 & l_2 y_2 & \dots & l_n y_n \\ l_1^2 x_1 & l_2^2 x_2 & \dots & l_n^2 x_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \qquad \mathcal{Y}_n = \begin{pmatrix} y_1 & y_2 & \dots & y_n \\ l_1 x_1 & l_2 x_2 & \dots & l_n x_n \\ l_1^2 y_1 & l_2^2 y_2 & \dots & l_n^2 y_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \tag{2}$$

There holds

$$\det_{1\leqslant i,j\leqslant n}\left(\frac{u_i y_j - v_i x_j}{l_j - k_i}\right) = \frac{1}{\prod_{i,j}(l_j - k_i)} \begin{vmatrix} \mathcal{U}_n & \mathcal{X}_n \\ \mathcal{V}_n & \mathcal{Y}_n \end{vmatrix}_{2n\times 2n} \tag{3}$$

Proof. *Let* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ *be* $n \times n$ *matrices, with* $\mathcal{A}$ *and* $\mathcal{C}$ *invertible. Using* $\left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{C} \end{smallmatrix}\right)\left(\begin{smallmatrix} I & \mathcal{A}^{-1}\mathcal{B} \\ I & \mathcal{C}^{-1}\mathcal{D} \end{smallmatrix}\right)$ *we obtain*

$$\begin{vmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{vmatrix} = |\mathcal{A}|\,|\mathcal{C}|\,|\mathcal{C}^{-1}\mathcal{D} - \mathcal{A}^{-1}\mathcal{B}| \tag{4}$$

where vertical bars denote determinants. Let $d(u) = diag(u_1, \dots, u_n)$ *and* $p_u = \prod_{1\leqslant i\leqslant n} u_i$. *We define similarly* $d(v)$, $d(x)$, $d(y)$ *and* p_v, p_x, p_y. *From the previous identity we get*

$$\begin{aligned} \begin{vmatrix} \mathcal{A}d(u) & \mathcal{B}d(x) \\ \mathcal{C}d(v) & \mathcal{D}d(y) \end{vmatrix} &= |\mathcal{A}|\,|\mathcal{C}|\; p_u p_v \left| d(v)^{-1}\mathcal{C}^{-1}\mathcal{D}d(y) - d(u)^{-1}\mathcal{A}^{-1}\mathcal{B}d(x) \right| \\ &= |\mathcal{A}|\,|\mathcal{C}| \left| d(u)\mathcal{C}^{-1}\mathcal{D}d(y) - d(v)\mathcal{A}^{-1}\mathcal{B}d(x) \right| \end{aligned} \tag{5}$$

The special case $\mathcal{A} = \mathcal{C}$, $\mathcal{B} = \mathcal{D}$, *gives*

$$\begin{vmatrix} \mathcal{A}d(u) & \mathcal{B}d(x) \\ \mathcal{A}d(v) & \mathcal{B}d(y) \end{vmatrix}_{2n\times 2n} = det(\mathcal{A})^2 \det_{1\leqslant i,j\leqslant n}\left((u_i y_j - v_i x_j)(\mathcal{A}^{-1}\mathcal{B})_{ij}\right) \tag{6}$$

Let $W(k)$ be the Vandermonde matrix with rows $(1\ldots 1)$, $(k_1\ldots k_n)$, $(k_1^2\ldots k_n^2)$, …, and $\Delta(k)=\det W(k)$ its determinant. Let

$$\mathcal{K}(t)=\prod_{1\leqslant m\leqslant n}(t-k_m) \tag{7}$$

and let C be the $n\times n$ matrix $(c_{im})_{1\leqslant i,m\leqslant n}$, where the c_{im}'s are defined by the partial fraction expansions:

$$1\leqslant i\leqslant n \qquad \frac{t^{i-1}}{\mathcal{K}(t)}=\sum_{1\leqslant m\leqslant n}\frac{c_{im}}{t-k_m} \tag{8}$$

We have the two matrix equations:

$$C=W(k)\,\mathrm{diag}(\mathcal{K}'(k_1)^{-1},\ldots,\mathcal{K}'(k_n)^{-1}) \tag{9a}$$

$$C\cdot\left(\frac{1}{l_j-k_m}\right)_{1\leqslant m,j\leqslant n}=W(l)\,\mathrm{diag}(\mathcal{K}(l_1)^{-1},\ldots,\mathcal{K}(l_n)^{-1}) \tag{9b}$$

This gives the (well-known) identity:

$$\left(\frac{1}{l_j-k_m}\right)_{1\leqslant m,j\leqslant n}=\mathrm{diag}(\mathcal{K}'(k_1),\ldots,\mathcal{K}'(k_n))W(k)^{-1}W(l)\,\mathrm{diag}(\mathcal{K}(l_1)^{-1},\ldots,\mathcal{K}(l_n)^{-1}) \tag{10}$$

We can thus rewrite the determinant we want to compute as:

$$\left|\frac{u_iy_j-v_ix_j}{l_j-k_i}\right|_{1\leqslant i,j\leqslant n}=\prod_m\mathcal{K}'(k_m)\prod_j\mathcal{K}(l_j)^{-1}\left|(u_iy_j-v_ix_j)(W(k)^{-1}W(l))_{ij}\right|_{n\times n} \tag{11}$$

We shall now make use of (6) with $A=W(k)$ and $B=W(l)$.

$$\begin{aligned}\left|\frac{u_iy_j-v_ix_j}{l_j-k_i}\right|_{1\leqslant i,j\leqslant n}&=\Delta(k)^{-2}\prod_m\mathcal{K}'(k_m)\prod_j\mathcal{K}(l_j)^{-1}\begin{vmatrix}W(k)d(u)&W(l)d(x)\\W(k)d(v)&W(l)d(y)\end{vmatrix}\\&=\frac{(-1)^{\frac{n(n-1)}{2}}}{\prod_{i,j}(l_j-k_i)}\begin{vmatrix}W(k)d(u)&W(l)d(x)\\W(k)d(v)&W(l)d(y)\end{vmatrix}_{2n\times 2n}\end{aligned} \tag{12}$$

The sign $(-1)^{n(n-1)/2}=(-1)^{[\frac{n}{2}]}$ is the signature of the permutation which exchanges rows i and $n+i$ for $i=2,4,\ldots,2[\frac{n}{2}]$ and transforms the determinant on the right-hand side into $\begin{vmatrix}U_n&X_n\\V_n&Y_n\end{vmatrix}$. This concludes the proof. □